Convergence properties of nonlinear functional link adaptive filters

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The analysis of some convergence properties of a recent class of nonlinear adaptive filters, known as ‘functional link adaptive filters’ (FLAFs) is aimed at. In particular, a convergence stability range is derived for nonlinear FLAFs using trigonometric series expansion.

Introduction: In recent years, a great interest in nonlinear adaptive filtering has arisen. Several methods have been investigated in order to model unknown nonlinear systems. Among them, prevalent techniques are based on the use of nonlinear transformations [1, 2]. Block-based Wiener-Hammerstein models using nonlinear transformations have also been investigated [3]. A new nonlinear adaptive filtering model based on functional links has been recently proposed [4, 5]. Such a functional link adaptive filter (FLAF) fruitfully combines the nonlinear modelling capabilities of functional links and the filtering properties of linear adaptive algorithms. This allows FLAFs to be computationally less expensive than artificial neural networks and adaptive Volterra filters, thus resulting as an effective tool to model nonlinearities in several applications.

In this Letter, we investigate some convergence properties of nonlinear FLAFs, using trigonometric series expansion. In particular, we derive an upper bound for the step-size parameter. Such a bound defines the range in which the convergence stability is guaranteed. To this end, a steady-state analysis is derived, providing the formulation of the ‘excess mean square error’ (EMSE) for a FLAF. Simulation results are shown to demonstrate the effectiveness of the proposed analysis.

Functional link adaptive filter: The FLAF approach is based on the idea of enhancing the input signal by representing it in a space of higher dimension. The FLAF architecture is composed of two main blocks: a nonlinear functional expansion of the input followed by a linear filtering, as depicted in Fig. 1. At the nth time step, the FLAF receives an input $\mathbf{x}_n \in \mathbb{R}^M = \left[x[n] \quad x[n-1] \quad \cdots \quad x[n-M+1]\right]^{\mathrm{T}}$, where M is the input buffer length, and processes it by means of a ‘functional expansion block’ (FEB). The FEB contains the set of Q ‘functional links’ $\Phi = \{\varphi_0(\cdot), \varphi_1(\cdot), \ldots, \varphi_{Q-1}(\cdot)\}$, which consists of a series of functions satisfying universal approximation constraints. The FEB processes the input buffer $\mathbf{x}_n$ by passing its ith element as argument for the chosen functions, each yielding a subvector $\bar{\mathbf{g}}_{i,n} \in \mathbb{R}^Q = \left[\varphi_0(x[n-i]) \quad \varphi_1(x[n-i]) \quad \cdots \quad \varphi_{Q-1}(x[n-i])\right]^{\mathrm{T}}$ $(i = 0, \ldots, M-1)$. The concatenation of subvectors $\bar{\mathbf{g}}_{i,n}$ yields the ‘expanded buffer’ $\mathbf{g}_n$

$$\begin{aligned}\mathbf{g}_n &= \left[\bar{\mathbf{g}}_{0,n}^{\mathrm{T}} \quad \bar{\mathbf{g}}_{1,n}^{\mathrm{T}} \quad \cdots \quad \bar{\mathbf{g}}_{M-1,n}^{\mathrm{T}}\right]^{\mathrm{T}} \\ &= \left[g_0[n] \quad g_1[n] \quad \cdots \quad g_{M_e-1}[n]\right]^{\mathrm{T}}\end{aligned} \tag{1}$$

where $g_m[n]$ $(m = 0, \ldots, M_e - 1)$ is the generic mth nonlinear element of the expanded buffer at the nth time instant and $M_e \geq M$ is the expanded buffer length. In this Letter, we consider a set of functional links based on a trigonometric series expansion in which all the used functions are nonlinear, thus resulting in a purely nonlinear expanded buffer

$$\varphi_j(x[n-i]) = \begin{cases} \sin(p\pi x[n-i]), & j = 2p-2 \\ \cos(p\pi x[n-i]), & j = 2p-1 \end{cases} \tag{2}$$

where $j = 0, \ldots, Q-1$ is the functional link index and $p = 1, \ldots, P$ is the expansion index, P being the ‘expansion order’. The functional link set described by (2) is composed of $Q = 2P$ functional links. Therefore the expanded buffer length is $M_e = QM = 2PM$.

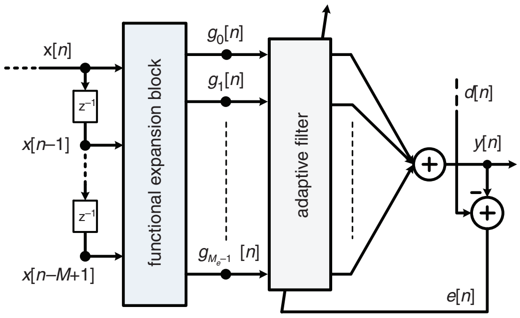

Fig. 1 *Functional link adaptive filter*

It is now possible to write the output of the nonlinear FLAF at nth time instant simply as the output of a FIR filter

$$y[n] = \mathbf{g}_n^{\mathrm{T}} \mathbf{w}_{n-1} \tag{3}$$

where $\mathbf{w}_n \in \mathbb{R}^{M_e} = \left[w_0[n] \quad w_1[n] \quad \ldots \quad w_{M_e-1}[n]\right]^{\mathrm{T}}$ is the coefficient vector of the nonlinear FLAF. Let

$$e[n] = d[n] - y[n] \tag{4}$$

be the *a priori* error of the proposed architecture, where $d[n]$ is the desired signal. Filter adaptation can be performed by minimising the overall square error $e^2[n]$ through a classical ‘least mean square’ (LMS) algorithm (see for example [6]):

$$\mathbf{w}_n = \mathbf{w}_{n-1} + \mu[n]\mathbf{g}_n e[n] \tag{5}$$

where $\mu[n]$ is the step-size parameter.

Convergence stability: To evaluate the convergence properties of the nonlinear FLAF, let us consider the *a posteriori* error, defined as

$$\begin{aligned}\varepsilon[n] &= d[n] - \mathbf{g}_n^{\mathrm{T}}\mathbf{w}_n = d[n] - \mathbf{g}_n^{\mathrm{T}}\left(\mathbf{w}_{n-1} + \mu[n]\mathbf{g}_n e[n]\right) \\ &= \left(1 - \mu[n]\|\mathbf{g}_n\|^2\right)e[n]\end{aligned} \tag{6}$$

To guarantee the convergence of the algorithm, the norm of the *a posteriori* error must be not greater than that of the *a priori* error [6], i.e. $|\varepsilon[n]| \leq |e[n]|$. Therefore from (6) it is possible to derive the following relation:

$$\left|1 - \mu[n]\|\mathbf{g}_n\|^2\right| \leq 1 \tag{7}$$

which implies the following stability range for choice of the step-size parameter $\mu[n]$:

$$0 < \mu[n] \leq \frac{2}{\|\mathbf{g}_n\|^2} \tag{8}$$

Now, recalling the expression of $\mathbf{g}_n$ in (1), it is easy to derive that

$$\|\mathbf{g}_n\|^2 = \mathbf{g}_n^{\mathrm{T}}\mathbf{g}_n = M_e/2 = PM \tag{9}$$

This is due to the trigonometric nature of the nonlinear expansion, since $\sin^2(\cdot) + \cos^2(\cdot) = 1$. Finally, the convergence stability range is

$$0 < \mu[n] \leq \frac{2}{PM} \tag{10}$$

It is very interesting to note that (10) does not depend on the nature of the input signal, but only on the expansion order P and on the input buffer length M.

Steady-state analysis: We now prove the effectiveness of the proposed stability range through a steady-state evaluation. To this end, an important performance measure is represented by the EMSE (see for example [6]) that is defined as the excess from the minimum value of the mean square error.

Let us denote the unknown system to identify with $\mathbf{w}_0$. The desired signal can be written as (see for example [6])

$$d[n] = \mathbf{g}_n^{\mathrm{T}}\mathbf{w}_0 + v[n] \tag{11}$$

where $v[n]$ is a zero-mean I.I.D. additive noise. Defining

$$\widetilde{\boldsymbol{w}}_n = \boldsymbol{w}_0 - \boldsymbol{w}_n \tag{12}$$

and taking into account (5), it is possible to write

$$\widetilde{\boldsymbol{w}}_n = \widetilde{\boldsymbol{w}}_{n-1} - \mu[n]\boldsymbol{g}_n e[n] \tag{13}$$

Evaluating the energies of both sides of (13), we obtain

$$\|\widetilde{\boldsymbol{w}}_n\|^2 = \|\widetilde{\boldsymbol{w}}_{n-1}\|^2 - 2\mu[n]\boldsymbol{g}_n^{\mathrm{T}}\widetilde{\boldsymbol{w}}_{n-1}e[n] + \mu^2[n]\|\boldsymbol{g}_n\|^2 e^2[n] \tag{14}$$

Considering (12) and (11), it is possible to rewrite the term $\boldsymbol{g}_n^{\mathrm{T}}\widetilde{\boldsymbol{w}}_{n-1}$ of (14) as

$$\boldsymbol{g}_n^{\mathrm{T}}\widetilde{\boldsymbol{w}}_{n-1} = \boldsymbol{g}_n^{\mathrm{T}}\boldsymbol{w}_0 - \boldsymbol{g}_n^{\mathrm{T}}\boldsymbol{w}_{n-1} = e[n] - v[n] = \bar{e}[n] \tag{15}$$

Moreover, taking the expectation of both sides of (14) and knowing that, at steady-state $\mathrm{E}\left\{\|\widetilde{\boldsymbol{w}}_n\|^2\right\} = \mathrm{E}\left\{\|\widetilde{\boldsymbol{w}}_{n-1}\|^2\right\}$ [6], it can be achieved

$$2\mathrm{E}\{e[n]\bar{e}[n]\} = \mu[n]\|\boldsymbol{g}_n\|^2\mathrm{E}\{e^2[n]\} \tag{16}$$

Expectations in (16) can be expressed in terms of $\bar{e}[n]$. Taking into account (4) and remembering that $v[n]$ is uncorrelated with $e[n]$, thus $\mathrm{E}\{e[n]v[n]\} = 0$, it is possible to write

$$\mathrm{E}\{e[n]\bar{e}[n]\} = \mathrm{E}\{(\bar{e}[n] + v[n])\bar{e}[n]\} = \mathrm{E}\{\bar{e}^2[n]\} \tag{17}$$

$$\mathrm{E}\{e^2[n]\} = \mathrm{E}\left\{(\bar{e}[n] + v[n])^2\right\} = \mathrm{E}\{\bar{e}^2[n]\} + \sigma_v^2 \tag{18}$$

where σ_v^2 is the variance of the noise $v[n]$. Replacing (17) and (18) in (16), it is possible to obtain

$$2\mathrm{E}\{\bar{e}^2[n]\} = \mu[n]\|\boldsymbol{g}_n\|^2(\mathrm{E}\{\bar{e}^2[n]\} + \sigma_v^2) \tag{19}$$

Denoting with $\zeta_{\mathrm{LMS}} = \lim_{n\to\infty}\mathrm{E}\{\bar{e}^2[n]\}$ the EMSE of the nonlinear FLAF adapted by the LMS algorithm, from (19) and using (9), we achieve

$$\zeta_{\mathrm{LMS}} = \frac{\mu[n]PM\sigma_v^2}{2 - \mu[n]PM} \tag{20}$$

It is possible to note from (20) that the expansion order P and the input buffer M directly affect the EMSE behaviour of a nonlinear FLAF using an LMS algorithm. For the consistency of ζ_{LMS} in (20), it is necessary that the EMSE is a positive value. This can be easily satisfied if (10) holds.

Experimental results: In this Section, we show a simple experimental test in order to prove the effectiveness of (10). We use a zero-mean Gaussian noise with unitary variance to identify a nonlinear system.

The system that must be identified is given by the following sixth-order 'moving average' system

$$H(z) = 0.1 - 0.2z^{-1} + 0.5z^{-2} - 0.3z^{-3} + 0.1z^{-4} - 0.05z^{-5} + 0.001z^{-6} \tag{21}$$

whose input is the signal

$$x[n] = \frac{2u[n]}{1 + |u[n]|^2} \tag{22}$$

In (22), $u[n]$ is obtained as

$$u[n] = au[n-1] + \sqrt{1-a^2}\,\xi[n] \tag{23}$$

where $\xi[n]$ is a zero-mean white Gaussian noise with unitary variance and $0 \le a < 1$ is a parameter that determines the level of correlation between adjacent samples. Fig. 2 shows the EMSE for a system identification test, using $a = 0.95$, an input buffer length $M = 8$ and $P = 6$. To verify the validity of (10), we compare four nonlinear FLAFs having different step-size values: $\mu_1 = 0.01 \ll 2/PM$, $\mu_2 = 0.04 < 2/PM$, $\mu_3 = 0.0416 = 2/PM$ and $\mu_4 = 0.042 > 2/PM$. Results are averaged over 1000 runs. Fig. 2 shows that the performance convergence is satisfactory when the step-size parameter is smaller than the proposed upper bound and that the algorithm does not converge anymore as soon as the step-size parameter exceeds such a bound (see the behaviour of the FLAF with μ_4). Therefore the results in Fig. 2 prove that the algorithm is able to converge if (10) is verified.

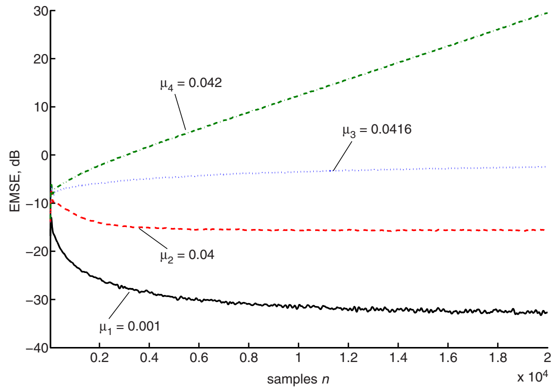

Fig. 2 *EMSE behaviour for proposed system identification test*

Conclusion: We have derived some interesting convergence results for the class of FLAFs. In particular, an upper bound for the choice of the step-size parameter and the expression of the EMSE are analytically achieved for nonlinear FLAFs, using trigonometric series expansion. Experimental results have proved the effectiveness of the proposed convergence stability range.

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One or more of the Figures in this Letter are available in colour online.

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